PROBABILISTIC ANALYSIS OF METHODS FOR NUMERICAL INTEGRATION

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The general formulation of the problem under consideration is usually given as follows:

Let $K \subset \mathbb{R}^n$ be a compact set, $X = X(K)$ a functional space over $K$ and $L : X \to \mathbb{R}$, linear functional. A well known and an important problem is to compute the value $L(x)$ for any $x \in X$. However, in general this problem is difficult since only for some special $L$ and $X$ one can find a convenient expression for $L(x)$. In some cases we can construct a sequence $\{L_n\}$ of linear approximators of $L$ obeying the following two properties:

i) For any $n \in \mathbb{N}$ and $x \in X$ the computation of $L_n(x)$ is relatively easy (e.g. if $L_n(x)$ is given by a quadrature formula).

ii) $L_n(x) \to L(x)$ as $n \to \infty$ for any $x \in X$.

Unfortunately, sometimes property ii) does not hold for all $x \in X$ and for that property additional conditions for $X$ are needed. Let us mention that in this area there is a problem called the stopping problem: Let $\varepsilon > 0$ and the integer $k$ be fixed. If $n$ is such that $|L_{n+k}(x) - L_n(x)| < \varepsilon$, then we stop the computation procedure and take either $L_n(x)$ or $L_{n+k}(x)$ as an $\varepsilon$-approximation of $L(x)$. It is impossible, however, to choose a stopping rule $n$ appropriate for all $x \in X$. In particular, if $M > 0$ and $n, k \in \mathbb{N}$ are fixed, it is possible to find $x \in X$ such that $L(x) = M$ but $L_n(x) = \cdots = L_{n+k}(x) = 0$. Obviously, the solution of the stopping problem depends on the numerical method, i.e. on the sequence of approximators $\{L_n\}$, and on the properties of the elements $x \in X$.

Let us turn now to the construction of a function which is "bad" for any method of numerical integration over $C[0,1]$.

Let $X = C[0,1]$, $L(x) = \int_0^1 x(t) \, dt$, $x \in C[0,1]$ and for $m = n, n+1, \ldots, n+k$, $\{L_n\}$ be a sequence of approximators for $L$, where $L_n(x) = \sum_{i=0}^{m} c_i^m x(t_i)$, $0 = t_0^m < t_1^m < t_2^m < \cdots < t_m^m = 1$, $c_i^m \geq 0$, $\sum_{i=1}^{m} c_i^m = 1$. Below $M > 0$ is a fixed number. We construct $x(t)$, $t \in [0,1]$ as follows:

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a) \( x(t) = 0 \) for \( t = t_j^m, j = 0, \ldots, m, m = n, n + 1, \ldots, n + k. \)

b) \( x(t) = qM \) for \( t = (t_i^m + t_j^m)/2 \) if there are no other points of interpolation in the subinterval \([t_i^m, t_j^m]\). 

c) \( x(t) \) is linear between these points.

Obviously, \( x \in C[0, 1] \) and \( L_m(x) = 0 \) for \( m = n, n + 1, \ldots, n + k. \) However, \( L(x) = qM/2 \) and if we choose \( q > 2 \) then \( L(x) > M \) and \( L(x) - L_m(x) > M, \) too.

Clearly, for the function \( x \) satisfying a), b), c), the stopping problem does not have a satisfactory solution. The same holds for polynomials approximating \( x \) (e.g. for Bernstein polynomials). Hence, more restrictions on \( X \) would eventually guarantee the smallness of the error in the stopping problem.

In any case, involving functions like the one described above, we can ask the following natural question: In the space \( X \), what is the portion of these “bad” functions, i.e. functions \( x \in X \), such that the computation of \( L(x) \) can be performed with an error greater than fixed \( \varepsilon, \varepsilon > 0? \) A correct answer can be given in the framework of a well defined probability model and the problem is reduced to finding the probability of the event \( \{|L(x) - L_n(x)| > \varepsilon\} \) for fixed \( \varepsilon > 0 \) and \( n. \)

If we know the probability \( P\{|L(x) - L_n(x)| < \varepsilon\}, \) we can compare different numerical methods, e.g. different sequences of approximators \( \{L_n\} \). Of a great importance is any case when the above probability is “close” to 1 for any \( \varepsilon > 0 \). One of the aims of the present study is to find the probability \( P\{|L(x) - L_n(x)| > \varepsilon\} \) for a given functional space \( X \), linear functional \( L \) and approximating sequence \( \{L_n\}. \) To reach that aim we have to pass the following three steps:

1) Specify the probability model \((X, F, P), \) i.e. we have to define the probability measure \( P \) on the \( \sigma \)-algebra \( F \) (usually \( F \) is the Borel \( \sigma \)-algebra generated by the open subsets of \( X) \).

2) Define the linear operator \( L \) and the sequence of approximators \( \{L_n\} \) as random variables on the probability space \((X, F, P) \).

3) Find, when possible, the distributions of the random variables \( L, L_n \) and of the error \( D_n = L - L_n. \)

If we know the distribution of \( D_n \), we can find the probability \( P\{|L - L_n > \varepsilon\} \) exactly, or find its upper bound. Moreover, we can choose a probabilistic criterion for comparing different methods and then looking for the best method with respect to this criterion. One possibility is to estimate the variance \( \text{Var}(D_n) \) of the random error \( D_n. \) This criterion, being a natural extension of the Gauss least squares method, is widely used in stochastics and its applications.

Let us focus our attention to the space \( C = C[0, 1] \) of all continuous, real valued functions \( x(t), t \in [0, 1] \) such that \( x(0) = 0. \) Let \( F^* \) denote the \( \sigma \)-algebra of Borel subsets of \( C, \) and \( P \) be the standard Wiener measure of \( F. \) The construction of the Wiener measure is well known and can be found in the literature, see e.g. Hida [1] or Lamperti [4]. The natural process, defined on the probability space \((C, F, P)\), denoted here by \( X_t, t \geq 0, \) is usually called a standard Wiener process.

It is well known that if \( L : C[0, 1] \rightarrow \mathbb{R} \) is a linear continuous functional then \( L(x) = \int_0^1 x(t) d\varphi(t) \) where \( \varphi \in V[0, 1] \) is a function of bounded variation (This is exactly the Riesz theorem, e.g. see Kolmogorov and Fomin [2]). Let now \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1, \) be a partition of \([0, 1]\) with a diameter \( d_n = \max_{1 \leq i < n} (t_i - t_{i-1}) \rightarrow 0 \) as \( n \rightarrow \infty \) and let

\[
T_n(x) = \sum_{i=1}^n x(t_i) [(\varphi(t_i) - \varphi(t_{i-1})]]
\]
Then \( T_n(x) \to L(x) \) for any \( x \in C[0,1] \) and therefore the distribution of \( L \) is a limit of the distributions of \( T_n \) as \( n \to \infty \). Thus the first step is to find the distribution of \( T_n \). An easy transformation shows that

\[
T_n = \sum_{i=1}^{n} X_{t_i}[(\varphi(t_i) - \varphi(t_{i-1}))] = \sum_{i=1}^{n} [\varphi(1) - \varphi(t_{i-1})] \cdot (X_{t_i} - X_{t_{i-1}}).
\]

The last quantity is a sum of independent and normally distributed random variables where \( X_{t_i} - X_{t_{i-1}} \sim N(0, t_i - t_{i-1}) \). Hence

\[
T_n \sim N(0, \sigma_n^2), \quad \text{where} \quad \sigma_n^2 = \sum_{i=1}^{n} (\varphi(1) - \varphi(t_{i-1}))^2 \cdot (t_i - t_{i-1}).
\]

Thus \( L \sim N(0, \sigma^2) \) with \( \sigma^2 = \lim \sigma_n^2 = \int_0^1 (\varphi(1) - \varphi(t))^2 \, dt, \, n \to \infty. \)

Hence, we have established the following result.

**Proposition 1.** Let \( L(x) = \int_0^1 x(t) \, d\varphi(t) \). Then the random variable \( L \) is normally distributed:

\[
L \sim N(0, \sigma^2), \quad \sigma^2 = \int_0^1 (\varphi(1) - \varphi(t))^2 \, dt.
\]

If \( \varphi(t) = t \), then \( L(x) = \int_0^1 x(t) \, dt \), and \( L \sim N(0, 1/3) \). Further on we consider in more details just this case. Let now \( \{L_n\} \) be a sequence of approximators of \( L \) and \( L_n(x) = \sum_{i=1}^{n} c_i \cdot x(t_i) \) with \( t_j \) are as above and \( c_i \geq 0, c_0 + c_1 + \cdots + c_n = 1 \). By the Riesz theorem \( L_n(x) = \int_0^1 x(t) \, d\varphi_n(t) \) where \( \varphi_n(t) \) is a step-wise function with jumps at the point \( t_i \) and size of the jumps proportional to \( c_i \). The error \( D_n(x) = L(x) - L_n(x) \) is also a linear functional and \( D_n(x) = \int_0^1 x(t) \, d(t - \varphi_n(t)) \). For \( D_n \) we obtain \( D_n \sim N(0, \delta_n^2) \) where \( \delta_n^2 = \int_0^1 (1 - \varphi_n(1) - t + \varphi_n(t))^2 \, dt \). In other words we have found the distribution of the error \( D_n \). Now we can compare the errors of using different quadrature formulas and look for the formula with the smallest variance of error. Since \( \text{Var}(D_n) = \delta_n^2 = \int_0^1 (1 - \varphi_n(1) - t + \varphi_n(t))^2 \, dt \), we want to minimize this quantity. The following transformations hold:

\[
\delta_n^2 = \int_0^1 (1 - t - (\varphi_n(1) - \varphi_n(t)))^2 \, dt
\]

\[
= \int_0^1 (1 - t)^2 \, dt - 2 \int_0^1 (1 - t)(\varphi_n(1) - \varphi_n(t)) \, dt + \int_0^1 (\varphi_n(1) - \varphi_n(t))^2 \, dt
\]

\[
= \frac{1}{3} - 2 \sum_{i=1}^{n} d_i \cdot \int_{t_{i-1}}^{t_i} (1 - t) \, dt + \sum_{i=1}^{n} d_i^2 \cdot \int_0^1 \, dt
\]

\[
= \frac{1}{3} + \sum_{i=1}^{n} [d_i^2(t_i - t_{i-1}) + d_i(t_i - t_{i-1})(t_i + t_{i-1} - 2)],
\]
where $d_i = \varphi_n(1) - \varphi_n(t_i)$ in subinterval $(t_{i-1}, t_i)$.

Thus we obtain

$$
\delta_n^2 \geq \frac{1}{3} - \frac{1}{4} \sum_{i=1}^{n} \left( t_i - t_{i-1} \right) \left( 2 - t_i - t_{i-1} \right)^2,
$$

where an equality holds only if $2d_i = 1 - t_i - t_{i-1}$. This observation and the restriction $0 = t_0 < t_1 < \cdots < t_n = 1$ imply that:

$$
t_i = \frac{i}{n}, \quad i = 0, 1, 2, \ldots, n, \quad c_0 = c_n = \frac{1}{2n}, \quad c_i = \frac{1}{n}, \quad i = 1, 2, \ldots, n-1, \quad \delta_n^2 = \frac{1}{(12n^2)}.
$$

The quadrature formula based on these $t_i$, $c_i$ corresponds to the well known trapezoidal method. Thus we have proved the following result.

**Proposition 2.** Among all the methods, the trapezoidal method has the smallest variance of error over the space $C[0, 1]$ with Wiener measure.

**Remark.** The above reasoning can be extended so to cover a case like $L(x) = \int_0^1 x(t) d\varphi(t)$ where $x$ is a more general Gaussian process.

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**REFERENCES**


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